Malaysian Journal of Mathematical Sciences 8(S): 163- 172 (2014)
Special Issue: International Conference on Mathematical Sciences and Statistics 2013 (ICMSS2013)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Integral Operators Preserving Univalence

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ABSTRACT

We introduce two new integral operators $F_{\alpha,\beta}$ and $H_{\alpha,\beta,\gamma}$ acting on the class of normalized analytic functions $\mathcal{A}$, where α,β and γ are complex parameters. Indeed, we derive sufficient conditions on the parameters α,β and γ to obtain that $F_{\alpha,\beta}(g)$ and $H_{\alpha,\beta,\gamma}(g)$ are univalent functions in the open unit disk $\mathbb{U}$, whenever g is univalent in $\mathbb{U}$.

Keywords: Univalent functions, univalence criteria, integral operators, preserving univalence.

1. INTRODUCTION AND PRELIMINARY

Let $\mathcal{A}$ be the class of functions analytic in the open unit disk $\mathbb{U}:=\{z:|z|<1\}$ and have the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n, \quad (z\in\mathbb{U}). \tag{1}$$

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions univalent (one-to-one) in $\mathbb{U}$. Ozaki and Nunokawa (1972) proved, for $g\in\mathcal{A}$ with $g(z)\neq 0$ in $0<|z|<1$, that the condition

$$\left|\frac{z^2 g'(z)}{g^2(z)}-1\right|\leq 1, \quad (z\in\mathbb{U}) \tag{2}$$

is sufficient for g to be in the class $\mathcal{S}$.
Let us introduce and consider the following integral operators defined on $\mathcal{A}$

by

$$F_{\alpha,\beta}(g)(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{(\alpha-1)g(u)}{\alpha u - g(u)}\right)^{\beta-1} \mathrm{d}u\right]^{1/\beta} \tag{3}$$

and

$$H_{\alpha,\beta,\gamma}(g)(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{\alpha u - g(u)}{(\alpha-1)g(u)}\right)^{1/\gamma} \mathrm{d}u\right]^{1/\beta}, \tag{4}$$

where $g \in \mathcal{A}$ with $g(z) \neq 0$ in $0 < |z| < 1$ and α, β, γ are certain complex numbers. In some occasions during the study of $H_{\alpha,\beta,\gamma}(g)$, the parameter γ cannot be chosen such that $\gamma = 1/(1-\beta)$ with $\beta \in \mathbb{R}$. We treat this case by considering the function $F_{\alpha,\beta}(g)$ independently with $H_{\alpha,\beta,\gamma}(g)$, for $\beta \in \mathbb{R}$ or $\mathbb{C}$.

Note that, for $f \in \mathcal{A}$:

(i) If we substitute $g(u) = \alpha u f(u)/[(\alpha-1)u + f(u)]$ in (3) where $f(u) \neq 0$ in $0 < |u| < 1$, then $F_{\alpha,\beta}(g)$ becomes

$$G_\beta(f)(z) = \left[\beta \int_0^z [f(u)]^{\beta-1} \mathrm{d}u\right]^{\frac{1}{\beta}}. \tag{5}$$

(ii) If we substitute $g(u) = \alpha u f'(u)/[f'(u) + \alpha - 1]$ in (3), where $f'(u) \neq 0$ in $0 < |u| < 1$, then $F_{\alpha,\beta}(g)$ becomes

$$I_\beta(f)(z) = \left[\beta \int_0^z [u f'(u)]^{\beta-1} \mathrm{d}u\right]^{\frac{1}{\beta}}. \tag{6}$$

(iii) If we substitute $g(u) = \alpha u^2 e^{f(u)}/[\alpha - 1 + u e^{f(u)}]$ in (3), then $F_{\alpha,\beta}(g)$ becomes

$$T_\beta(f)(z) = \left[\beta \int_0^z \left[u^2 e^{f(u)}\right]^{\beta-1} \mathrm{d}u\right]^{\frac{1}{\beta}}. \tag{7}$$

(iv) If we substitute $g(u) = \alpha u^2/[(\alpha-1)f(u) + u]$ in (4), then $H_{\alpha,\beta,\gamma}(g)$ becomes

$$Q_{\beta,\gamma}(f)(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{f(u)}{u}\right)^{\frac{1}{\gamma}} \mathrm{d}u\right]^{\frac{1}{\beta}}. \tag{8}$$

(v) If we substitute $g(u) = \alpha u^2/[(\alpha-1)f(u) + u]$, $\beta = 1$ and $\delta = 1/\gamma$

in (4), then $H_{\alpha,\beta,\gamma}(g)$ becomes

$$W_{\delta}(f)(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\delta} \, \mathrm{d}u. \tag{9}$$

For the functions in $\mathcal{A}$ which are satisfying (2) and more general for the functions of $\mathcal{S}$, the problem of preserving univalence under the above integral operators (i-v) has been studied by many authors including Pescar (2003, 2005, 2006, 2006A), Breez and Breez (2003, 2004) and Kim and Merkes (1972).

In this article, we study the univalence of $F_{\alpha,\beta}(g)$ and $H_{\alpha,\beta,\gamma}(g)$ for the functions g of the general class $\mathcal{S}$. Namely, we derive sufficient conditions on the parameters α, β and γ to obtain that $F_{\alpha,\beta}(g)$ and $H_{\alpha,\beta,\gamma}(g)$ are members of $\mathcal{S}$, whenever $g \in \mathcal{S}$.

To prove our main results, we need the following theorem:

Theorem 1.1. 1(Pascu (1987)). *Let* $\gamma \in \mathbb{C}$, $Re\gamma > 0$ *and* $f \in \mathcal{A}$. *If*

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zf''(z)}{f'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, then for all $\beta \in \mathbb{C}$, $\mathrm{Re}\beta \ge \mathrm{Re}\gamma$, the function

$$G_{\beta}(z) = \left[\beta \int_0^z t^{\beta-1} f'(t) \, \mathrm{d}t\right]^{1/\beta}$$

is univalent in $\mathbb{U}$.

2. MAIN RESULTS

Let us prove the following theorem:

Theorem 2.1. 2*Let* $g \in \mathcal{S}$ *with* $g(z) \neq 0$ *for* $0 < |z| < 1$. *If* $\alpha \in \mathbb{C}$ *with* $0 < |\alpha| < 1/4$ *and*

$$|1-\beta| \le \frac{1-4|\alpha|}{16|\alpha|}\mathrm{Re}\beta, \quad \text{for } \mathrm{Re}\beta \in (0,1) \tag{10}$$

or

$$|1-\beta| \leq \frac{1-4|\alpha|}{16|\alpha|}, \quad \text{for } \operatorname{Re}\beta \in [1,\infty), \tag{11}$$

then the function $F_{\alpha,\beta}(g)$ *defined by (3) belongs to* $\mathcal{S}$.

Proof. In view of (3), the function $F_{\alpha,\beta}(g)$ can be rewritten as

$$F_{\alpha,\beta}(g)(z) = \left[\beta\left(\frac{\alpha}{\alpha-1}\right)^{1-\beta}\int_0^z u^{\beta-1}\left(\frac{u}{g(u)}-\frac{1}{\alpha}\right)^{1-\beta}\mathrm{d}u\right]^{\frac{1}{\beta}}. \tag{12}$$

Let us consider the function

$$f(z) = \left(\frac{\alpha}{\alpha-1}\right)^{1-\beta}\int_0^z \left(\frac{u}{g(u)}-\frac{1}{\alpha}\right)^{1-\beta}\mathrm{d}u. \tag{13}$$

We can choose regular branch of the function $z/g(z)$ to be equal to 1 at the origin. Hence the function f is regular in $\mathbb{U}$ and $f(0) = 1 - f'(0) = 0$, which means $f \in \mathcal{A}$. A simple computation shows that

$$F_{\alpha,\beta}^{\beta-1}(g)(z)\cdot F'_{\alpha,\beta}(g)(z) = z^{\beta-1}f'(z). \tag{14}$$

Therefore, $F_{\alpha,\beta}(g)(0) = 1 - F'_{\alpha,\beta}(g)(0) = 0$ and hence $F_{\alpha,\beta}(g) \in \mathcal{A}$. Because $g \in \mathcal{S}$, we have

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \frac{1+|z|}{1-|z|} \tag{15}$$

and

$$|g(z)| \geq \frac{|z|}{(1+|z|)^2} \tag{16}$$

for all $z \in \mathbb{U}$. Also, by computations, we get

$$f'(z) = \left(\frac{\alpha}{\alpha-1}\right)^{1-\beta}\left(\frac{z}{g(z)}-\frac{1}{\alpha}\right)^{1-\beta},$$

$$f''(z) = (1-\beta)\left(\frac{\alpha}{\alpha-1}\right)^{1-\beta}\left(\frac{z}{g(z)}-\frac{1}{\alpha}\right)^{-\beta}\left(\frac{g(z)-zg'(z)}{g^2(z)}\right)$$

and

$$\left|\frac{zf''(z)}{f'(z)}\right| = |1-\beta|\left|\frac{\alpha z}{\alpha z - g(z)}\right|\left|1-\frac{zg'(z)}{g(z)}\right|, \tag{17}$$

for all $z \in \mathbb{U}$. From (16) and (15), we have for $0 < r = |z| < 1$ and $0 < |\alpha| < 1/4$,

$$\left|\frac{\alpha z}{g(z)-\alpha z}\right| \leq \frac{|\alpha| r}{|g(z)|-|\alpha| r} \leq \frac{|\alpha|}{\frac{1}{(1+r)^2}-|\alpha|} \leq \frac{4|\alpha|}{1-4|\alpha|} \tag{18}$$

and

$$\left|1-\frac{zg'(z)}{g(z)}\right| \leq 1+\left|\frac{zg'(z)}{g(z)}\right| \leq \frac{2}{1-r}. \tag{19}$$

Next, for $0 < Re\beta < 1$, the function

$$t: (0,1) \to \mathbb{R}, \quad t(x) = 1 - r^{2x}, \quad (0 < r < 1)$$

is an increasing function and for $|z| = r,\ z \in \mathbb{U}$, we obtain

$$1 - |z|^{2\mathrm{Re}\beta} \leq 1 - r^2, \tag{20}$$

for all $z \in \mathbb{U}$. Hence, from (17), (18), (19) and (20), we obtain

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{16|\alpha||1-\beta|}{(1-4|\alpha|)\mathrm{Re}\beta}. \tag{21}$$

Combining (21) with condition (10), we get

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad \mathrm{Re}\beta \in (0,1), \tag{22}$$

for all $z \in \mathbb{U}$. Now, for $\mathrm{Re}\beta \geq 1$, we observe that the function

$$s: [1,\infty) \to \mathbb{R}, \quad s(x) = \frac{1-r^{2x}}{x}, \quad (0 < r < 1)$$

is a decreasing function and for $r = |z|,\ z \in \mathbb{U}$, we have

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \leq 1 - r^2, \tag{23}$$

for all $z \in \mathbb{U}$. Hence, from (17), (18), (19) and (23), we get

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{16|\alpha||1-\beta|}{1-4|\alpha|}. \tag{24}$$

Combining (24) with condition (11), we arrive at

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \quad \mathrm{Re}\beta \in [1,\infty), \tag{25}$$

for all $z \in \mathbb{U}$. Since

$$f'(z) = \left(\frac{\alpha z - g(z)}{(\alpha-1)g(z)}\right)^{1-\beta}.$$

Then, applying (22) and (25) to Theorem 1.1 for $\beta = \gamma$, we establish that the function $F_{\alpha,\beta}(g)$ defined by (3) belongs to $\mathcal{S}$.

Assuming that β is real in Theorem 2.1 gives what follows:

Corollary 2.2. 3*Let* $g \in \mathcal{S}$ *with* $g(z) \neq 0$ *for* $0 < |z| < 1$. *If* $\alpha \in \mathbb{C}$ *with* $0 < |\alpha| < 1/4$ *and*

$$\beta \in \left[\frac{16|\alpha|}{12|\alpha|+1}, \frac{12|\alpha|+1}{16|\alpha|}\right],$$

then the function $F_{\alpha,\beta}(g)$ defined by (3) belongs to $\mathcal{S}$.

Proof. For $\beta \in (0,1)$, condition (10) yields

$$1-\beta \le \frac{1-4|\alpha|}{16|\alpha|}\beta$$

and hence the domain of β is reduced to

$$\beta \in \left[\frac{16|\alpha|}{12|\alpha|+1}, 1\right).$$

For $\beta \in [1,\infty)$, condition (11) yields

$$\beta - 1 \le \frac{1-4|\alpha|}{16|\alpha|}$$

and hence the domain of β is reduced to

$$\beta \in \left[1, \frac{12|\alpha| + 1}{16|\alpha|}\right].$$

Thus the result follows by applying Theorem 2.1 for the choice β is real.

The univalence of $H_{\alpha,\beta,\gamma}(g)$ is studied in the following theorem:

Theorem 2.3.4 *Let* $g \in \mathcal{S}$ *with* $g(z) \neq 0$ *when* $0 < |z| < 1$. *For* $\alpha \in \mathbb{C}$ *with* $0 < |\alpha| < 1/4$ *and* $Re\beta \geq Re\gamma$, *if*

$$|\gamma| \geq \frac{16|\alpha|}{\mathrm{Re}\gamma(1-4|\alpha|)}, \quad \text{when } \mathrm{Re}\gamma \in (0,1) \tag{26}$$

or

$$|\gamma| \geq \frac{16|\alpha|}{1-4|\alpha|}, \quad \text{when } \mathrm{Re}\gamma \in [1,\infty), \tag{27}$$

then the function $H_{\alpha,\beta,\gamma}(g)$ defined by (4) belongs to $\mathcal{S}$.

Proof. Consider the function

$$f(z) = (\alpha - 1)^{-\frac{1}{\gamma}} \int_0^z \left(\frac{\alpha u}{g(u)} - 1\right)^{\frac{1}{\gamma}} \mathrm{d}u. \tag{28}$$

The function $f \in \mathcal{A}$ because as $g \in \mathcal{S}$, we can choose a regular branch of the function $z/g(z)$ to be equal to 1 at the origin. Then a simple computation shows that

$$H^{\beta-1}_{\alpha,\beta,\gamma}(g)(z) \cdot H'_{\alpha,\beta,\gamma}(g)(z) = z^{\beta-1} f'(z). \tag{29}$$

Therefore, $H_{\alpha,\beta,\gamma}(g)(0) = 1 - H'_{\alpha,\beta,\gamma}(g)(0) = 0$ and so $H_{\alpha,\beta,\gamma}(g) \in \mathcal{A}$. Also we have

$$f'(z) = (\alpha - 1)^{-\frac{1}{\gamma}} \left(\frac{\alpha z}{g(z)} - 1\right)^{\frac{1}{\gamma}},$$

$$f''(z) = (\alpha - 1)^{-\frac{1}{\gamma}} \left(\frac{\alpha z}{g(z)} - 1\right)^{\frac{1}{\gamma}-1} \left(\frac{\alpha g(z) - \alpha z g'(z)}{\gamma g^2(z)}\right).$$

This yields

$$\left|\frac{zf''(z)}{f'(z)}\right| \quad \leq \quad \frac{1}{|\gamma|}\left|\frac{\alpha z}{\alpha z-g(z)}\right|\left|1-\frac{zg'(z)}{g(z)}\right| \tag{30}$$

for all $z \in \mathbb{U}$. Combining (30) with (18) and (19), we obtain for $0 < r = |z| < 1$ and $0 < |\alpha| < 1/4$,

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{|\gamma|} \cdot \frac{4|\alpha|}{1-4|\alpha|} \cdot \frac{2}{1-r}. \tag{31}$$

If $0 < Re\gamma < 1$, then from (20) and (31), we have $1-|z|^{2\mathrm{Re}\gamma} \leq 1-|z|^2$ and

$$\frac{1-|z|^{2\mathrm{Re}\,\gamma}}{\mathrm{Re}\,\gamma}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{|\gamma|\mathrm{Re}\,\gamma} \cdot \frac{16|\alpha|}{1-4|\alpha|}. \tag{32}$$

Combining (32) with condition (26), we obtain

$$\frac{1-|z|^{2\mathrm{Re}\,\gamma}}{\mathrm{Re}\,\gamma}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad \mathrm{Re}\gamma \in (0,1), \tag{33}$$

for all $z \in \mathbb{U}$. If $\mathrm{Re}\gamma \geq 1$, then from (23) and (31), we have $1-|z|^{2\mathrm{Re}\gamma} \leq (1-|z|^2)\mathrm{Re}\gamma$ and

$$\frac{1-|z|^{2\mathrm{Re}\,\gamma}}{\mathrm{Re}\,\gamma}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{|\gamma|} \cdot \frac{16|\alpha|}{1-4|\alpha|}, \tag{34}$$

for all $z \in \mathbb{U}$. Combining (34) with condition (27), we obtain

$$\frac{1-|z|^{2\mathrm{Re}\,\gamma}}{\mathrm{Re}\,\gamma}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad \mathrm{Re}\gamma \in [1,\infty), \tag{35}$$

for all $z \in \mathbb{U}$. Since

$$f'(z) = \left(\frac{\alpha z - g(z)}{(\alpha-1)g(z)}\right)^{\frac{1}{\gamma}}.$$

Then, applying (33) and (35) to Theorem 1.1 with $\mathrm{Re}\beta \geq \mathrm{Re}\gamma$, we establish that the function $H_{\alpha,\beta,\gamma}(g)$ defined by (4) belongs to $\mathcal{S}$.

Assuming that β and γ are real in Theorem 2.3 with $\beta = \gamma$ gives what follows:

Corollary 2.4. 5*Let* $g \in \mathcal{S}$ *with* $g(z) \neq 0$ *for* $0 < |z| < 1$. *If* $\alpha \in \mathbb{C}$ *with* $0 < |\alpha| < 1/4$ *and*

$$\gamma \in \left[\min\left\{1, \frac{4\sqrt{|\alpha|}}{\sqrt{1-4|\alpha|}}\right\}, 1\right] \bigcup \left[\max\left\{1, \frac{16|\alpha|}{1-4|\alpha|}\right\}, \infty\right),$$

then the function $H_{\alpha,\gamma,\gamma}(g)$ belongs to $\mathcal{S}$.

Proof. From conditions (26) and (27), we have for $\gamma \in (0,1]$,

$$\gamma^2 \geq \frac{16|\alpha|}{1-4|\alpha|}$$

and hence the domain of γ is reduced to

$$\gamma \in \left[\min\left\{1, \frac{4\sqrt{|\alpha|}}{\sqrt{1-4|\alpha|}}\right\}, 1\right].$$

For $\gamma \in [1, \infty)$, condition (27) yields

$$\gamma \in \left[\max\left\{1, \frac{16|\alpha|}{1-4|\alpha|}\right\}, \infty\right).$$

Thus the result follows by applying Theorem 2.3 for the choice β and γ are real with $\beta = \gamma$.

ACKNOWLEDGEMENT

This work is funded by the Deanship of Scientific Research in Zarqa University, Jordan.

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